

A gauge field theory of fermionic Continuous-Spin Particles

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Abstract

In this letter, we suggest a local covariant action for a gauge field theory of fermionic Continuous-Spin Particles (CSPs). The action is invariant under gauge transformations without any constraint on both the gauge field and the gauge transformation parameter. The Fang-Fronsdal equations for a tower of massless fields with all half-integer spins arise as a particular limit of the equation of motion of fermionic CSPs.

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1. Introduction

The unitary representations of the Poincaré group in four spacetime dimensions were first examined by E. Wigner in [1]. For massless particles, there is a class of representations, the so-called “continuous-spin” particles, for which the eigenstates of different helicities are mixed under Lorentz transformations, similarly to the class of massive particles. In 3+1 dimensions, there exists only two types of CSP: the bosonic case where the spectrum of eigenvalues of the helicity operator is all the integers, and the fermionic case where the spectrum span all the half-integers. The helicity is defined, more covariantly, as $W^2|h\rangle = -\rho^2|h\rangle$

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where h is the helicity, W^μ is the Pauli-Lubanski vector and the real parameter ρ (with the dimension of a mass) determines the degree of mixing of eigenstates. The eigenstates can be labeled by either integer or half-integer eigenvalues h , depending on the representation type. In the $\rho \rightarrow 0$ limit, the helicity-eigenstates reduce to the familiar ones that are Lorentz invariant, in the sense that they do not mix under Lorentz boosts (see e.g. [2] for more review). Recently, it was argued that CSPs might evade the Weinberg no-go theorem on covariant soft emission amplitudes and could thus mediate long-range interactions [2].

As first pointed out by A.M. Khan and P. Ramond in [3], one suggestive way to think about a CSP is as the limit of a massive particle where its mass m goes to zero while its spin s goes to infinity with their product being fixed ($m \rightarrow 0$ and $s \rightarrow \infty$, with $ms = \rho$). This group-theoretical observation was translated at the field-theoretical level in [4] where Fronsda-like equations of motion for bosonic CSPs and Fang-Fronsda-like ones for fermionic CSPs were obtained from the above limit of the corresponding equations [5, 6] for massive higher-spin particles (see e.g. [7] for a review) and shown to be equivalent to Wigner's equations [8] (see also [9] for more details).

More recently, P. Schuster and N. Toro presented a local covariant action for bosonic CSPs, formulated with the help of an auxiliary Lorentz vector η^μ localized to the unit hyperboloid $\eta^2 = -1$ [10]. This localization on a hyperboloid improved their initial proposal [11] and allows to recover precisely the equations of [4] as Euler-Lagrange equations. See also the recent analysis of V. O. Rivelles [12].

Until now, the gauge field theory of fermionic CSPs was missing from the literature at the level of the action. To describe supersymmetric CSP multiplets [13] or cross-interactions between bosonic and fermionic CSPs, it is unavoidable to construct an action of this type. The layout of the letter is as follows. In section 2, a local covariant action of fermionic CSPs is proposed and we elaborate on its different aspects. In section 3, taking $\rho = 0$, Fang-Fronsda equations for half-integer helicities will be obtained [6]. We conclude and present open problems in section 4.

We will work in the “mostly minus” signature and focus on spacetime dimension four but the higher ($D \geq 4$) and lower¹ dimensional generalizations are straightforward.

2. Local and covariant action

We propose an action for the free fermionic CSPs as

$$S_{free} = \int d^4x d^4\eta \left[\delta'(\eta^2 + 1) \bar{\Psi} (\gamma \cdot \eta - i)(\gamma \cdot \partial_x) \Psi + \delta(\eta^2 + 1) \bar{\Psi} \Delta \Psi \right], \quad (1)$$

¹There is a version in three spacetime dimensions of CSPs which can be thought as a massless generalization of anyons [14].

where γ^μ are gamma matrices, $\delta'(a) = \frac{d}{da}\delta(a)$ and $\Delta = \partial_\eta \cdot \partial_x + \rho$.

The gauge field $\Psi(\eta, x)$ is a spinor field, of which the spinor index has been omitted. It is assumed that Ψ is analytic in η^μ . From the action, it is clear that $\Psi(\eta, x)$ has mass dimension $3/2$, as it should. When $\rho = 0$, the helicity eigenstates factorize into a tower of states with half-integer eigenvalues. The action is written in an enlarged spacetime where inhomogeneous Lorentz transformations act on x^μ ($x' = \Lambda x + a$) and homogeneous Lorentz transformations act on an auxiliary 4-vector coordinate η^μ ($\eta' = \Lambda \eta$). The delta functions in (1) illustrate that the η dependence of $\Psi(\eta, x)$ is localized to a unit hyperboloid in η -space, an internal space that encodes spin. Note that no dynamics is carried out in η -space. The action is invariant under the gauge transformation

$$\begin{aligned} \delta\Psi(\eta, x) = & \left[(\gamma \cdot \partial_x)(\gamma \cdot \eta + i) - (\eta^2 + 1)\Delta \right] \epsilon(\eta, x) \\ & + (\eta^2 + 1)(\gamma \cdot \eta - i)\chi(\eta, x), \end{aligned} \quad (2)$$

where $\epsilon(\eta, x)$ and $\chi(\eta, x)$ are arbitrary spinor gauge transformation parameters and there is no constraint on them. The χ symmetry is the analogue of the one in [10, 12] which allows us to remove the triple gamma-trace part of the gauge field.

In the presence of background currents, linear interactions can be given by

$$\begin{aligned} S_{int} = & -i \int d^4x d^4\eta \delta'(\eta^2 + 1) \left[\bar{\Psi}(\eta, x)(\gamma \cdot \eta - i)\sigma(\eta, x) \right. \\ & \left. - \bar{\sigma}(\eta, x)(\gamma \cdot \eta + i)\Psi(\eta, x) \right], \end{aligned} \quad (3)$$

where σ and $\bar{\sigma}$ are spinor sources.

The gauge invariance of S_{int} leads to two continuity-like condition

$$\left[\delta(\eta^2 + 1)(\gamma \cdot \eta - i)\Delta \right] \sigma(\eta, x) = 0, \quad (4)$$

$$\bar{\sigma}(\eta, x) \left[\overleftarrow{\Delta} \delta(\eta^2 + 1)(\gamma \cdot \eta + i) \right] = 0, \quad (5)$$

for each source, where $\overleftarrow{\Delta}$ means that Δ operates to the left. Using (1) and (3), it is straightforward to obtain a covariant equation of motion for the field Ψ

$$\begin{aligned} & \left[\delta'(\eta^2 + 1)(\gamma \cdot \eta - i)(\gamma \cdot \partial_x) + \delta(\eta^2 + 1)\Delta \right] \Psi \\ & = i \delta'(\eta^2 + 1)(\gamma \cdot \eta - i)\sigma. \end{aligned} \quad (6)$$

As will be shown in the next section, this equation of motion describes a single fermionic CSP. In this approach, there is no constraint on the gauge field, contrarily to the Fang-Fronsdal formulation (see [15] for a local unconstrained formulation of massless higher-spin fields).

One of the main purposes of this paper is to present a local and covariant action for the fermionic CSPs which reproduces fermionic higher-spin massless particles in the $\rho \rightarrow 0$ limit (called “helicity correspondence” in [10]). To

demonstrate this connection, we shall transform our equations to those in ω -space, the conjugate space of the η -space. In ω -space, we will show that our equation of motion is equivalent to the Fang-Fronsdal-like equation [4], which was obtained from the massive Fang-Fronsdal equation and is equivalent to the Wigner equations [8].

3. Relation to the Fang-Fronsdal equation

We perform a Fourier transformation in η^μ to express the Grassmann variables in the ω -space as

$$\Psi(\omega, x) \equiv \int d^4\eta e^{i\eta \cdot \omega} \delta'(\eta^2 + 1)(\gamma \cdot \eta + i)\Psi(\eta, x), \quad (7)$$

$$\sigma(\omega, x) \equiv \int d^4\eta e^{i\eta \cdot \omega} \delta'(\eta^2 + 1)(\gamma \cdot \eta - i)\sigma(\eta, x), \quad (8)$$

$$\epsilon(\omega, x) \equiv \int d^4\eta e^{i\eta \cdot \omega} \delta(\eta^2 + 1)(\gamma \cdot \eta + i)\epsilon(\eta, x). \quad (9)$$

Notice that the fields in the left-hand-sides are unconstrained while the ones in the right-hand-side are constrained. More precisely, the equations (7) and (9) can be understood as the general solutions of the triple gamma-trace condition

$$(\gamma \cdot \partial_\omega + 1)(\partial_\omega \cdot \partial_\omega - 1)\Psi(\omega, x) = 0, \quad (10)$$

and the gamma-trace condition

$$(\gamma \cdot \partial_\omega + 1)\epsilon(\omega, x) = 0, \quad (11)$$

which are equivalent to the ones in [4] (up to a multiplication by the matrix $i\gamma^5$ as explained below). Let us point out that the fields in the left-hand-sides of (7) and (9) do not uniquely determine the fields $\Psi(\eta, x)$ and $\epsilon(\eta, x)$ in the right-hand-side, but only up to some gamma-trace terms. In particular, the arbitrariness in the field $\Psi(\eta, x)$ is nothing but the χ symmetry in (2). In other words, the field $\Psi(\omega, x)$ is not affected by the χ symmetry. As one can check, the change of variables (7)-(9) converts some of the gauge symmetries of the original fields (e.g. the χ symmetry) into conditions imposed on the new fields (e.g. gamma-trace constraint). This fact is closely related to the standard conversion of first-class constraints into second-class ones.

Multiplying the equation (2) by $\delta'(\eta^2 + 1)(\gamma \cdot \eta + i)$ to the left, we obtain

$$\delta'(\eta^2 + 1)(\gamma \cdot \eta + i)\delta\Psi(\eta, x) = \Delta[\delta(\eta^2 + 1)(\gamma \cdot \eta + i)\epsilon(\eta, x)]. \quad (12)$$

Now, Fourier transforming (12) over the auxiliary variable η , the gauge transformation takes the form of

$$\delta\Psi(\omega, x) = (\omega \cdot \partial_x + i\rho)\epsilon(\omega, x), \quad (13)$$

where a constant factor has been absorbed in the gauge field. This is exactly the gauge transformation of the Fang-Fronsdal-like equation, proposed in [4]. Let us stress that the χ symmetry is absent in (13).

The continuity condition (4) appears in ω -space as

$$\left[(\gamma \cdot \partial_\omega + 1)(\gamma \cdot \partial_x) - (\omega \cdot \partial_x + i\rho)(\partial_\omega^2 - 1) \right] \sigma(\omega, x) = 0. \quad (14)$$

The equation of motion (6) turns into

$$i \left[(\gamma \cdot \partial_x) - (\omega \cdot \partial_x + i\rho)(\gamma \cdot \partial_\omega + 1) \right] \Psi(\omega, x) = \sigma(\omega, x). \quad (15)$$

A gauge invariant equation of motion equivalent to (15), with $\sigma = 0$, was obtained in [4] from the massless high-spin limit of the equation for fermionic massive particles, but no action leading to this equation of motion was presented. To see the equivalence between (15) and the equation written in [4], we can multiply (15) by the matrix $i\gamma^5$ to the left (with $\sigma = 0$) and get

$$\left[(\Gamma \cdot \partial_x) - (\omega \cdot \partial_x + i\rho)(\Gamma \cdot \partial_\omega + i\Gamma^5) \right] \Psi(\omega, x) = 0, \quad (16)$$

where $\Gamma^\mu = i\gamma^5\gamma^\mu$ and $\Gamma^5 = \gamma^5$. These new matrices Γ 's satisfy the same Clifford algebra as the original matrices γ 's and the obtained equation is the one in [4].²

Via a gauge-fixing procedure similar to the one in [4], one can show that the equation (15) without source describes a single fermionic CSP. In fact, we can impose the gauge

$$(\gamma \cdot \partial_\omega + 1)\Psi(\omega, x) = 0, \quad (17)$$

and get from (15) with $\sigma = 0$:

$$i(\gamma \cdot \partial_x)\Psi(\omega, x) = 0. \quad (18)$$

In turn, the equations (17) and (18) imply that

$$(\partial_\omega \cdot \partial_x)\Psi(\omega, x) = 0. \quad (19)$$

As explained in [4], these three equations (17)-(19) are equivalent to Wigner's equations [8] which are known to describe a single fermionic CSP.

To make contact between the above equations and the corresponding Fang-Fronsdal equations, one can first rescale³ the auxilliary variable (and the gauge parameter) as follows: $\omega \rightarrow \rho^{\frac{1}{2}}\omega$ in (13)-(15) and then put $\rho = 0$. For instance, (15) reads in terms of the rescaled variable as

$$i \left[(\gamma \cdot \partial_x) - (\omega \cdot \partial_x + i\rho^{\frac{1}{2}})(\gamma \cdot \partial_\omega + \rho^{\frac{1}{2}}) \right] \Psi(\omega, x) = \sigma(\omega, x), \quad (20)$$

²Notice that the mostly plus signature was used in [4] and is responsible for a distinct i factor.

³X. B. is grateful to J. Mourad for discussions on the corresponding rescaling in the bosonic case.

which in the $\rho \rightarrow 0$ limit leads to the Fang-Fronsdal equation

$$i \left[(\gamma \cdot \partial_x) - (\omega \cdot \partial_x)(\gamma \cdot \partial_\omega) \right] \Psi(\omega, x) = \sigma(\omega, x), \quad (21)$$

Similarly, one gets from (14) in the same $\rho \rightarrow 0$ limit

$$\left[(\gamma \cdot \partial_\omega)(\gamma \cdot \partial_x) - (\omega \cdot \partial_x) \partial_\omega^2 \right] \sigma(\omega, x) = 0. \quad (22)$$

The spinor field Ψ can be considered of the form

$$\Psi(\omega, x) = \psi(x) + \omega^\mu \psi_\mu(x) + \frac{1}{2} \omega^\mu \omega^\nu \psi_{\mu\nu}(x) + \dots, \quad (23)$$

where ψ is a spinor (Dirac) field of helicity $\frac{1}{2}$, ψ_μ is a vector-spinor (Rarita-Schwinger) field of helicity $\frac{3}{2}$, $\psi_{\mu\nu}$ is a symmetric tensor-spinor field of helicity $\frac{5}{2}$, etc. We will have the same definition for the spinor field σ as above. For ϵ we can write

$$\epsilon(\omega, x) = \epsilon(x) + \omega^\mu \epsilon_\mu(x) + \frac{1}{2} \omega^\mu \omega^\nu \epsilon_{\mu\nu}(x) + \dots, \quad (24)$$

where ϵ is the gauge parameter of the helicity $\frac{3}{2}$ gauge field and so on. By assuming the fields analytic in ω -space, the Fang-Fronsdal formulation of half-integer spin gauge fields can conveniently be elaborated as follows:

According to (13) at $\rho = 0$, one can see that the Dirac field is not a gauge field, but all other massless fields transform under the gauge symmetries. The gauge transformations for $s = \frac{3}{2}, \frac{5}{2}, \dots$, take the standard form [6]

$$\begin{aligned} \delta\psi_\mu &= \partial_\mu \epsilon, \\ \delta\psi_{\mu\nu} &= \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu, \\ &\vdots \end{aligned} \quad (25)$$

The equation of motion (21), for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ reduce to

$$\begin{aligned} i(\gamma \cdot \partial)\psi &= \sigma, \\ i\left((\gamma \cdot \partial)\psi_\alpha - \partial_\alpha \psi'\right) &= \sigma_\alpha, \\ i\left((\gamma \cdot \partial)\psi_{\alpha\beta} - \partial_\alpha \psi'_\beta - \partial_\beta \psi'_\alpha\right) &= \sigma_{\alpha\beta}, \\ &\vdots \end{aligned}$$

which are exactly Fang-Fronsdal equations for half-integer higher-spin gauge fields [6]. The Fang-Fronsdal notation for trace has been used (γ -trace $\epsilon'_{\nu\rho\dots} \equiv \gamma^\mu \epsilon_{\mu\nu\rho\dots}$).

Ultimately, the continuity conditions (for $s = \frac{3}{2}, \frac{5}{2}, \dots$) can be extracted from (22)

$$\begin{aligned} \partial^\mu \sigma_\mu &= \frac{1}{2} \gamma^\mu \partial_\mu \sigma', \\ \partial^\nu \sigma_{\mu\nu} &= \frac{1}{2} (\gamma^\nu \partial_\nu \sigma'_\mu + \partial_\mu \sigma''), \\ &\vdots \end{aligned} \quad (26)$$

which indeed correspond to the ones in [6].

4. Conclusions and Discussion

In this letter, we proposed a local, covariant and gauge-invariant, action (1) to describe fermionic CSPs. As is standard in higher-spin literature, an auxiliary Minkowski space (η -space here) was used to encode spinning degrees of freedom. However, there is no dynamics within the η -space. We rewrote, in the conjugate ω -space, the gauge symmetries and the equation of motion, and related them to the ones in [4]. Finally, taking a suitable $\rho \rightarrow 0$ limit, the Fang-Fronsdal equations for fermionic higher spin gauge fields were correctly obtained.

The fermionic CSP action proposed here, together with the bosonic CSP action of Schuster and Toro action, may open a new window to probe supersymmetric CSPs, or investigate Yukawa-like interactions of CSPs. We let the canonical and path integral quantizations of fermionic CSPs for future work.

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